What happens to wavepackets of fermions when scattered by the Maldacena-Ludwig wall?

Keita Tsuji

Kavli Institute for the Physics and Mathematics of the Universe

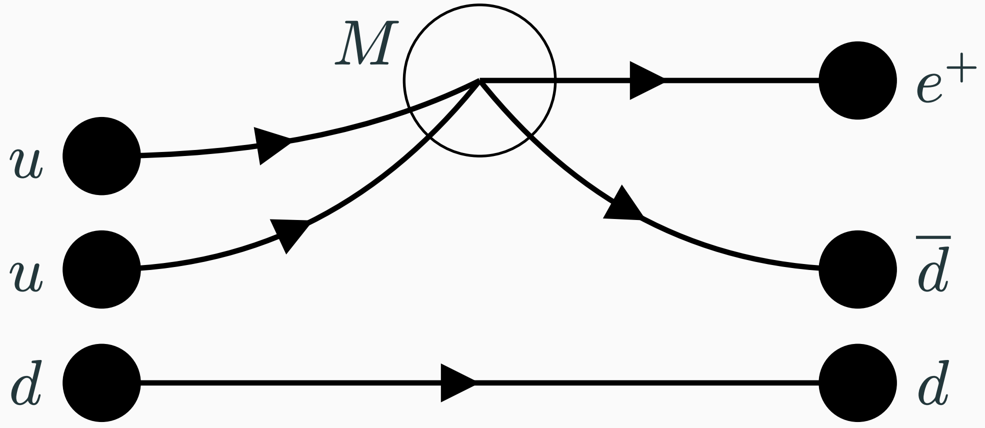
keita.tsuji@ipmu.jp

Based on [Tachikawa–KT–Watanabe 2603.25508]

Motivation: Callan–Rubakov problem

In the 1980s, Callan and Rubakov found that a monopole can catalyze proton decay [Callan 1983, Rubakov 1982]:

$$M + p \to M + e^+ + \text{hadrons.}$$



During this scattering process, "exotic" states that have fractional charges arise. This is the Callan–Rubakov problem.

Motivation: Callan–Rubakov problem

For simplicity, **we ignore gauge field fluctuations below**.

In a static monopole background $\boldsymbol{A}$ with a magnetic charge $g \in \mathbb{Z}$, the conserved angular momentum of a fermion with electric charge $q \in \mathbb{Z}$ is

$$\boldsymbol{J} = \boldsymbol{r} \times (\boldsymbol{p} - q\boldsymbol{A}) + \boldsymbol{S} - \frac{qg}{2}\hat{\boldsymbol{r}},$$

where the monopole is at $r = 0$. The minimum angular momentum is

$$j_0 = \frac{|qg| - 1}{2},$$

and there are $|qg|$ eigenvalues of J_z: $j_z = -j_0, -j_0 + 1, ..., j_0$.

Motivation: Callan–Rubakov problem

The probability density of a fermion near the core of the monopole $r \sim 0$ is

$$r^2|\Psi|^2 \propto r^{2\nu}, \quad \nu = \sqrt{(j-j_0)(j+j_0+1)}.$$

Only the lowest angular momentum mode $j = j_0$ reaches the core of the monopole, so we can ignore the $j > j_0$ modes.

For simplicity, we will consider the simplest case: $|q| = g = 1$. Then $j_0 = 0$ and this approximation is called the s-wave approximation. For the $j = 0$ s-wave mode,

$$0 = \boldsymbol{J} \cdot \hat{\boldsymbol{r}} = \boldsymbol{S} \cdot \hat{\boldsymbol{r}} - q/2,$$

so the radial component of the spin is fixed to be $\boldsymbol{S} \cdot \hat{\boldsymbol{r}} = q/2$.

Motivation: Callan–Rubakov problem

Since the radial component of the spin is fixed to be $\boldsymbol{S} \cdot \hat{\boldsymbol{r}} = q/2$, if $q > 0$, incoming fermions have helicity $-1/2$ and outgoing fermions have helicity $+1/2$.

For a massless fermion, helicity and chirality coincide, so the chirality and whether the mode is incoming or outgoing are correlated:

$U(1)$ charge q	chirality	s-wave
$+$	left-handed	incoming
$+$	right-handed	outgoing
$-$	left-handed	outgoing
$-$	right-handed	incoming

Motivation: Callan–Rubakov problem

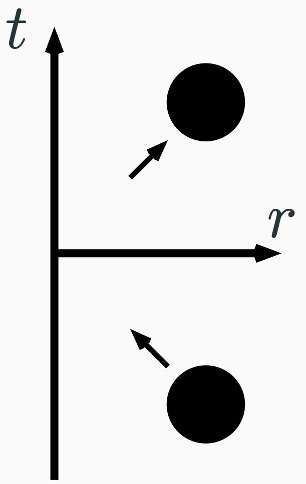
Let us consider a simple case: 4d $U(1)$ gauge theory with N_f massless Dirac fermions ψ with $q = 1$, which has an $SU(N_f)$ symmetry acting left-handed Weyl fermions ψ_L, ψ_R^c diagonally:

	$U(1)$	chirality	$SU(N_f)$	s-wave
ψ_L	$+1$	left-handed	$\boldsymbol{N_f}$	incoming
ψ_R^c	-1	left-handed	$\boldsymbol{N_f}$	outgoing
ψ_L^c	-1	right-handed	$\overline{\boldsymbol{N_f}}$	incoming
ψ_R	$+1$	right-handed	$\overline{\boldsymbol{N_f}}$	outgoing

The scattering of the s-waves appears unable to preserve both $U(1)$ and $SU(N_f)$ symmetries if $N_f \geq 3$. This is the Callan–Rubakov problem.

Background: Maldacena–Ludwig boundary condition

Even though the particle content indicates that the $SU(N_f)$ symmetry cannot be preserved, the 2d theory of the s-wave fermions can have an $SU(N_f)$-preserving conformal boundary condition.



We take the r-axis to the right and the t-axis upward. The correspondence of 4d and 2d fermions is:

4d s-wave	2d
incoming	left-movers
outgoing	right-movers

Background: Maldacena–Ludwig boundary condition

The 2d theory is a boundary CFT of N_f free massless Dirac fermions.
$SU(N_f)$ is vector-like in this theory, so it has no 't Hooft anomaly.

The relationship of a boundary condition and an 't Hooft anomaly is complicated.

Existence of a symmetry-preserving simple conformal boundary condition $\xrightarrow{\checkmark}$ No 't Hooft anomaly

Existence of a symmetry-preserving simple conformal boundary condition $\xleftarrow{???}$ No 't Hooft anomaly

[Wei–Zheng 2025] deals with counterexamples of the $\leftarrow$ direction.

Thus, the absence of an 't Hooft anomaly only suggests that there is a symmetry-preserving simple conformal boundary condition.

Background: Maldacena–Ludwig boundary condition

For $N_f = 4$, there is a boundary condition called the Maldacena–Ludwig boundary condition preserving the $U(1) \times SU(4)$ symmetry [Maldacena–Ludwig, 1995].

It is convenient to focus on the left-movers. We flip the $U(1)$ charge of the right-movers so that all symmetries are vector-like.

4d	2d	original $U(1)$	diagonal $U(1)$	$SU(4)$
incoming ψ_L	left-mover ψ	$+1$	$+1$	$\mathbf{4}$
outgoing ψ_R^c	right-mover $\tilde{\psi}$	-1	$+1$	$\mathbf{4}$

Thus the desired boundary condition needs to flip the diagonal $U(1)$.

Background: Maldacena–Ludwig boundary condition

The construction of the Maldacena–Ludwig boundary condition uses the famous $SO(8)$ triality.

$$\text{CFT of 4 massless Dirac fermions} \cong \text{CFT of 8 massless Majorana fermions},$$

so the theory has an $SO(8)$ symmetry, in which $SU(4)$ is embedded. The $\mathfrak{so}(8)_1$ currents are

$$:\overline{\psi}^i \psi^j:, \quad :\psi^i \psi^j: \ (i < j), \quad :\overline{\psi}^i \overline{\psi}^j: \ (i < j),$$

where $:\overline{\psi}^i \psi^j:$ are the $U(4) \cong U(1) \times SU(4)$ currents, the diagonal $U(1)$ current is $:\overline{\psi}^1 \psi^1: + :\overline{\psi}^2 \psi^2: + :\overline{\psi}^3 \psi^3: + :\overline{\psi}^4 \psi^4:$.

The primary states of the current algebra $\mathfrak{so}(8)_1$ are the following.

sectors		primary states	#(primary states)
vacuum sector	χ_0	$\|0\rangle_{\mathrm{NS}}$	1
vector sector	χ_V	$\psi^i_{-1/2}\|0\rangle_{\mathrm{NS}}, \overline{\psi}^i_{-1/2}\|0\rangle_{\mathrm{NS}}$	8
spinor sector	χ_S	$\|{+}{+}{+}{-}\rangle_{\mathrm{R}}, \|{+}{-}{-}{-}\rangle_{\mathrm{R}}, \ldots$	8
cospinor sector	χ_C	$\|{+}{+}{+}{+}\rangle_{\mathrm{R}}, \|{+}{+}{-}{-}\rangle_{\mathrm{R}}, \ldots$	8

Here, $|{\pm}{\pm}{\pm}{\pm}\rangle_{\mathrm{R}}$ denotes the ground states of the Ramond sector. For every two chiral Majorana fermions, the ground states are doubly degenerate, so there are 16 ground states in the CFT of 8 chiral Majorana fermions.

The numbers of the primary states of χ_V, χ_S, χ_C are the same. This reflects the triality of $SO(8)$.

Background: Maldacena–Ludwig boundary condition

χ_V and χ_S can be exchanged via bosonization:

$$\psi^i \sim :\exp(2\pi i \phi^i):, \quad \overline{\psi}^i \sim :\exp(-2\pi i \phi^i):, \ldots,$$

$$J^i = :\overline{\psi}^i \psi^i: \sim \partial \phi^i, \quad i = 1, 2, 3, 4,$$

where $\phi^1, \phi^2, \phi^3, \phi^4$ are 4 compact massless bosons. The primary fields are

sectors		primary fields
vacuum sector	χ_0	1
vector sector	χ_V	$\exp(\pm 2\pi i \phi^i)$
spinor sector	χ_S	$\exp(\pi i(\pm\phi^1 \pm \phi^2 \pm \phi^3 \pm \phi^4))$ with odd #(−)
cospinor sector	χ_C	$\exp(\pi i(\pm\phi^1 \pm \phi^2 \pm \phi^3 \pm \phi^4))$ with even #(−)

Background: Maldacena–Ludwig boundary condition

We change a basis of the bosons:

$$\begin{pmatrix} \tilde{\phi}^1 \\ \tilde{\phi}^2 \\ \tilde{\phi}^3 \\ \tilde{\phi}^4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{pmatrix}.$$

$\tilde{\phi}^1$ corresponds to the diagonal $U(1)$ current $J_1 + J_2 + J_3 + J_4$, so we flip the sign of $\tilde{\phi}^1$ on the boundary:

$$\tilde{\phi}^1_L = -\tilde{\phi}^1_R, \quad \tilde{\phi}^2_L = \tilde{\phi}^2_R, \quad \tilde{\phi}^3_L = \tilde{\phi}^3_R, \quad \tilde{\phi}^4_L = \tilde{\phi}^4_R$$

This is the Maldacena–Ludwig boundary condition.

Background: Maldacena–Ludwig boundary condition

In the original basis, the Maldacena–Ludwig boundary condition is

$$\phi_L^i = R^{ij}\phi_R^j\big|_{\text{boundary}}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad R^\mathsf{T} = R, \quad R^2 = 1.$$

This flips the diagonal $U(1)$, and one can check that all the $SU(4)$ currents are preserved. A vector-sector fermion ψ_L^1 turns into

$$\psi_L^1 \sim \,:\exp\big(2\pi i\phi_L^1\big): \xrightarrow{R} \,:\exp\Big(\pi i\big(\phi_R^1 - \phi_R^2 - \phi_R^3 - \phi_R^4\big)\Big):,$$

which is a spinor-sector excitation, which has a fractional $SO(8)$ charge.

Background summary

- In the 4d N_f massless fermions theory with a monopole, the scattering of the s-waves seems to be prohibited by the $SU(N_f) \times U(1)$ symmetry.
- However, in the 2d CFT of the s-waves, there can be a symmetry-preserving conformal boundary condition.
- For $N_f = 4$, the Maldacena–Ludwig boundary condition is such a boundary condition.
- The Maldacena–Ludwig boundary condition exchanges χ_V and χ_S, with χ_0 and χ_C fixed.
- The Maldacena–Ludwig solution of the Callan–Rubakov problem is that an incoming fermion in χ_V turns into an excitation in χ_S.

What is the remaining problem?

On a circle S^1, χ_S is in the Ramond sector, so the scattered state cannot be written by the original NS-sector fermions.

However, a two-fermion state in the NS sector is in χ_0, so after scattering, it still remains in χ_0. We can write it by the original NS-sector fermions.

We found the explicit expression of the scattered state:

$$|\Psi\rangle = \exp\left(\oint \frac{dw_1}{2\pi i} \oint \frac{dw_2}{2\pi i} D_{<,<}(w_1, w_2) \overline{\psi}_{\geq}(w_1) \psi_{\geq}(w_2) \right) |0\rangle,$$

where the meanings of the symbols will be explained later.

What can we learn from the explicit expression of the state?

From the explicit expression, we can extract a lot of information:

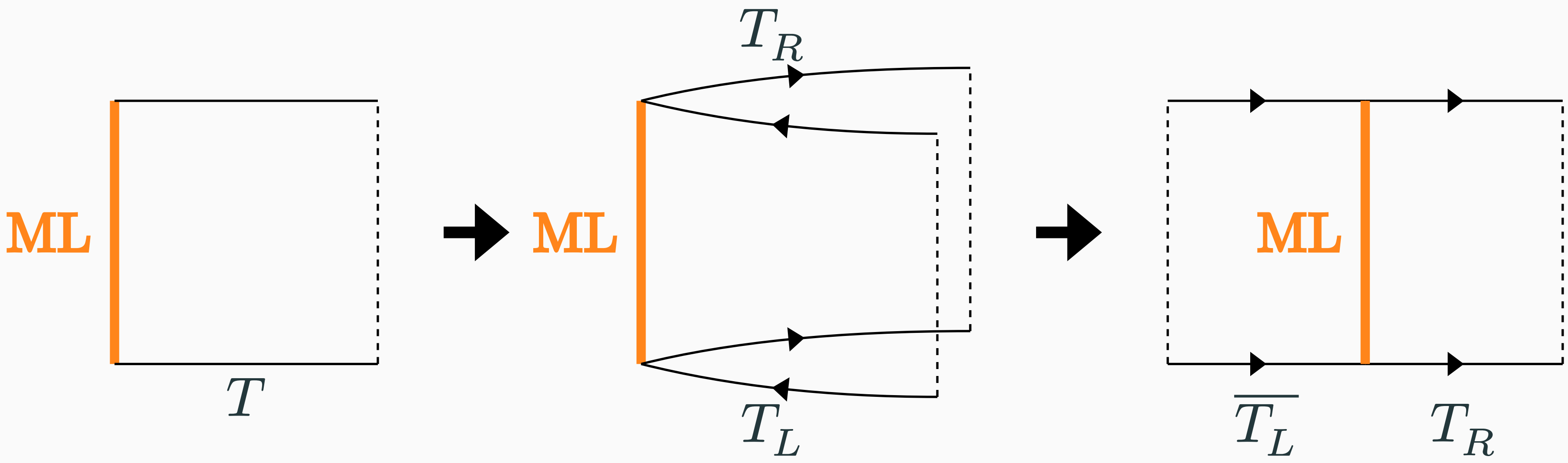
1. A two-fermion state $\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle$ turns into the state $|\Psi\rangle$, which can be written in terms of the original NS-sector fermions.
2. The energy density is conserved and the charge density remains localized at x_1, x_2.
3. The local integral of the charge density at $x = x_1$ is fractional:

$$(Q^1, Q^2, Q^3, Q^4) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

4. In the transformed state $|\Psi\rangle$, the average number of original fermions diverges.

Unfolding

Since the Majorana fermions are massless, the bulk theory is just a tensor product of left-movers and right-movers: $T = T_L \otimes T_R$. Then, we can unfold the theory:



The ML boundary becomes the ML wall after unfolding.

Unfolding

The ML boundary condition satisfies $J_L = RJ_R$ on the boundary, and the energy momentum tensor is

$$T = \frac{1}{2}\sum_{i=1}^{4} :J^i J^i: .$$

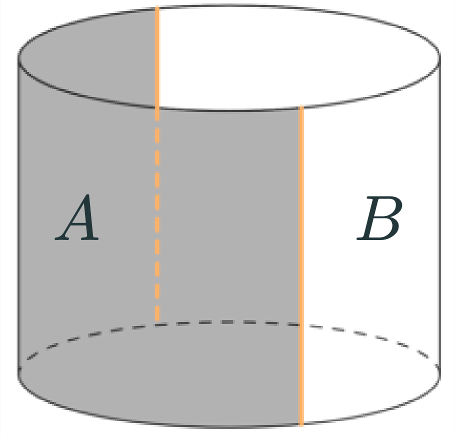
Since R is an orthogonal matrix, $T_L = T_R$ on the boundary. Thus, after unfolding, T is continuous on the ML wall. This means that the ML wall is not only conformal but also topological.

Actually, we can regard the ML wall as a non-invertible symmetry of the CFT of 8 chiral Majorana fermions.

Fusion

We can derive the fusion rule of the ML wall by considering a cylinder with two ML walls. Since $R^2 = 1$, the field content is

- χ_V in the area A, χ_S in the area B.
- χ_S in the area A, χ_V in the area B.
- χ_0, χ_C.



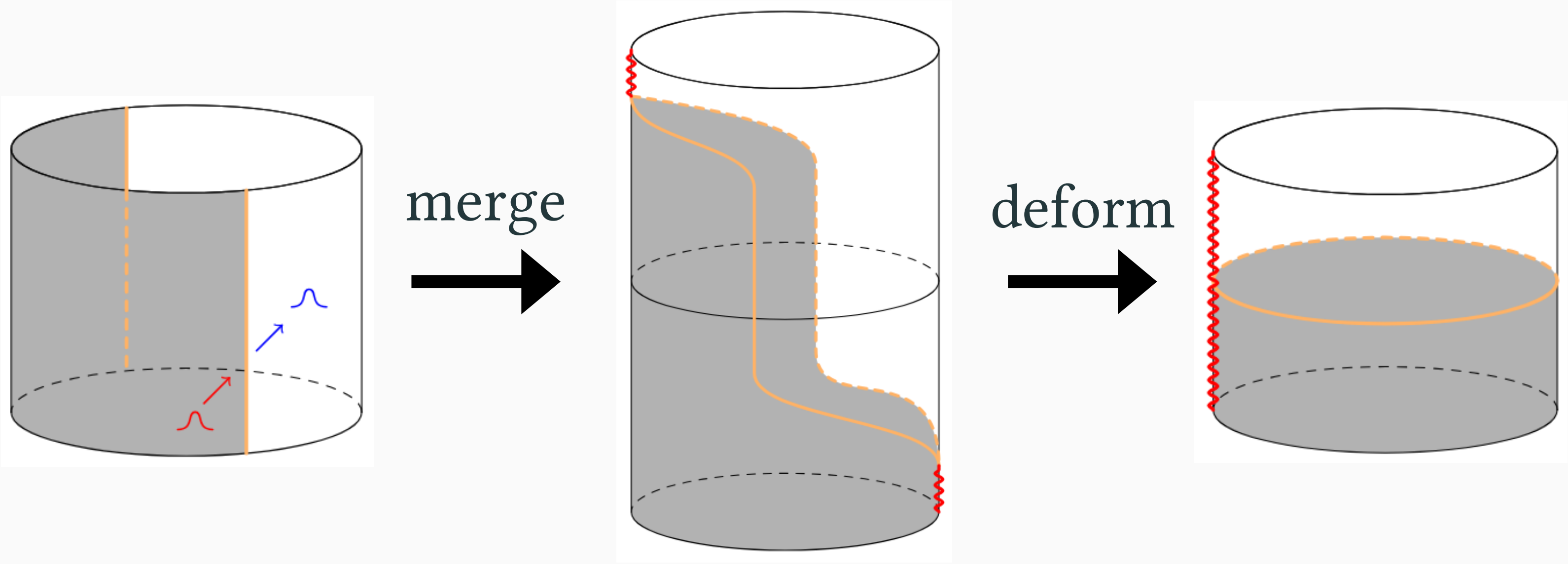
Then we shrink the area A, and the field content is just $\chi_0, \chi_V, \chi_S, \chi_C$ in the area B. The total Hilbert space is the direct sum $\mathcal{H}_{\mathrm{NS}} \oplus \mathcal{H}_{\mathrm{R}}$, so

$$(\text{ML wall}) \otimes (\text{ML wall}) = 1 \oplus (-1)^F,$$

where $(-1)^F$ is the chiral fermion parity.

Deform ML walls

When we merge two ML walls, they become $1 \oplus (-1)^F$ represented by the red lines. We can deform the ML wall so that it wraps around the spatial direction. Because of locality, the effect of the ML wall on a wavepacket is equivalent to the ML wall wrapping around the spatial direction.

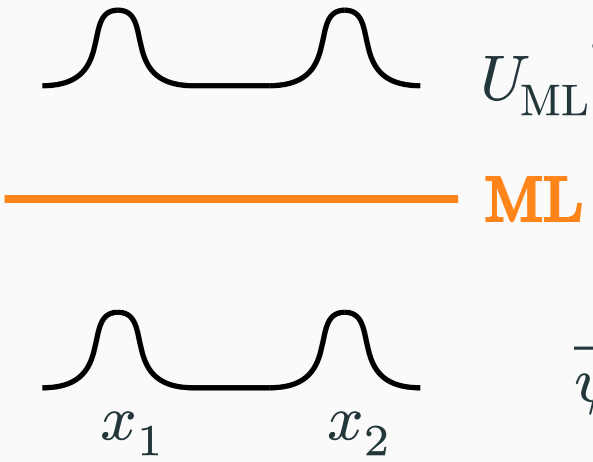


How to calculate the action of the ML wall

The ML boundary condition is $J_L^i = R^{ij} J_R^j$, so the ML wall wrapping in the spatial direction acts on the currents as

$$U_{\mathrm{ML}} J^i(x) U_{\mathrm{ML}}^{-1} = R^{ij} J^j(x),$$

so the strategy for calculating the action on a wavepacket is the following:



$U_{\mathrm{ML}} \overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle$ ← ③ (expression of $R^{ij} J^j(x)$)

↑ ② action of ML wall

$\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle$ → ① (expression of $J^i(x)$)

First step: write a wavepacket in terms of currents

For simplicity, we will do the computation on S^1 with radius 1.

Actually, it is easy to write a wavepacket in terms of $J^i(x)$. By the bosonization formulae

$$J^i(x) \sim \partial\phi^i(x), \quad \psi^i(x) \sim :\exp(2\pi i\phi^i(x)):, \quad \overline{\psi}^i(x) \sim :\exp(-2\pi i\phi^i(x)):,$$

the two-fermion state is

$$\frac{\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle}{\langle 0|\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle} = :\exp\left(2\pi i\int_0^{2\pi} dx c(x) J^1(x)\right):|0\rangle,$$

where $c(x)$ is a step function: $c(x) = \begin{cases} 1, & (x_1 < x < x_2) \\ 0, & (\text{otherwise}) \end{cases}.$

Second step: action of ML wall

$$U_{\text{ML}}\left(\frac{\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle}{\langle 0|\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle}\right)$$

$$= U_{\text{ML}}\left(:\exp\left(2\pi i\int_0^{2\pi} dx c(x) J^1(x)\right):|0\rangle\right)$$

$$= :\exp\left(2\pi i\int_0^{2\pi} dx c(x)\frac{1}{2}(J^1(x)-J^2(x)-J^3(x)-J^4(x))\right):|0\rangle.$$

The currents with different flavor indices commute, so the only thing left to do is to calculate $:\exp(\int dx f(x) J(x)):|0\rangle$ for an arbitrary function $f(x)$.

Computation of wavepacket

We move to radial quantization via $z = e^{ix}$.

It is enough to consider one complex fermion, so we drop the flavor index and just write $\psi(z)$. What we want to calculate is

$$:e^{iA_0}:|0\rangle, \quad A_0 = \oint_{|z|=1} \frac{dz}{2\pi i} f(z) J(z).$$

We will set $f(z) = \frac{1}{2} c(z)$ later.

We define “positive” and “negative” modes of f by

$$f(z) = \sum_n f_{-n} z^n, \quad f_{\geq}(z) = \sum_{n \geq 0} f_{-n} z^n, \quad f_{<}(z) = \sum_{n<0} f_{-n} z^n.$$

Computation of wavepacket

For fermions and currents, we define positive modes by

$$\psi_{\geq}(z) = \sum_{n\geq 0} \psi_{-n-1/2} z^n, \quad J_{\geq}(z) = \sum_{n\geq 0} J_{-n-1} z^n,$$

so the positive modes are creation operators. We have

$$:e^{iA_0}:|0\rangle = e^{iA}|0\rangle, \quad A = \oint_{|z|=1} \frac{dz}{2\pi i} f_{<}(z) J_{\geq}(z).$$

$J_{\geq}$ is fermion bilinear. Thus, $e^{iA}|0\rangle$ is a Gaussian state, which is completely determined by two-particle components and Wick's theorem.

Computation of wavepacket

Since $e^{iA}|0\rangle$ is a Gaussian state, we can write

$$e^{iA}|0\rangle = \exp\left(\oint_{|w_1|=1} \frac{dw_1}{2\pi i} \oint_{|w_2|=1} \frac{dw_2}{2\pi i} D_{<,<}(w_1, w_2)\overline{\psi}_{\geq}(w_1)\psi_{\geq}(w_2)\right)|0\rangle.$$

for the two-particle component $D_{<,<}(w_1, w_2)$, which can be evaluated by the two-particle component of the left-hand side as

$$D_{<,<}(z_1, z_2) = \frac{1}{z_2 - z_1}\left[\frac{\exp(2\pi i f_<(z_2))}{\exp(2\pi i f_<(z_1))} - 1\right]$$

by using the BCH formula $e^{-iA}\psi(z)e^{iA} = \exp(-2\pi i f_<(z))\psi(z)$.

Computation of wavepacket

We want to set $f(x)$ to be the step function $c(x)$ from x_1 to x_2, but then the norm of the state diverges. Thus we regulate it by shifting

$$\exp\bigl(ix_{1,2}\bigr) \to \zeta_{1,2} = \exp\bigl(-\varepsilon + ix_{1,2}\bigr), \quad \varepsilon > 0.$$

$\varepsilon > 0$ is a width of wavepackets localized around $z_{1,2} = \exp\bigl(ix_{1,2}\bigr)$.

Now we have $2\pi i f_<(z) = \dfrac{1}{2}\log\dfrac{z-\zeta_2}{z-\zeta_1}$, and the transformed state is

$$\exp\left(\oint\frac{dw_1}{2\pi i}\oint\frac{dw_2}{2\pi i}\frac{1}{w_2-w_1}\left(\sqrt{\frac{w_1-\zeta_1}{w_1-\zeta_2}\frac{w_2-\zeta_2}{w_2-\zeta_1}}-1\right)\overline{\psi}_\geq(w_1)\psi_\geq(w_2)\right)|0\rangle,$$

which is the desired expression.

What can we learn from the computation?

For the transformed state $|\Psi\rangle = U_{\mathrm{ML}}\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle$, we can say the following:

1. Since the ML wall acts on J^i by R^{ij} locally, in the transformed state $|\Psi\rangle$, the energy density and the charge density remain localized at x_1, x_2.
2. The local integral of the charge density at $x = x_1$ is fractional:

$$(Q^1, Q^2, Q^3, Q^4) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

3. The transformed state $|\Psi\rangle$ actually exists in the Fock space of NS-sector fermions $\psi^i, \overline{\psi}^i$ as expected.

However, these results follow from the action of U_{ML} on J^i.

Average number of particles

Using the explicit expression of $|\Psi\rangle$, we can study more subtle details.

We can prove that the average number $\langle N\rangle$ of particles (+ antiparticles) diverges in the limit $\varepsilon \to 0$.

Let us write the transformed state as

$$|\Psi\rangle = \exp\left(\sum_{n,m\geq 1} D_{nm}\overline{\psi}_{-n+1/2}\psi_{-m+1/2}\right)|0\rangle,$$

where D_{nm} is the coefficient of $\overline{\psi}_{-n+1/2}\psi_{-m+1/2}|0\rangle$.

Average number of particles

We regard D_{nm} as a matrix and denote the singular values of D as

$$s_1 \geq s_2 \geq \cdots \geq 0.$$

Then we can decompose the state into independent fermion modes:

$$|\Psi\rangle = \exp\left(\sum_{n=1}^{\infty} s_n a_n^\dagger b_n^\dagger\right)|0\rangle = \prod_{n=1}^{\infty}\left(1 + s_n a_n^\dagger b_n^\dagger\right)|0\rangle.$$

Then the singular value s_n can be interpreted as the amplitude of the n-th mode.

Average number of particles

The norm and the average number of particles of $|\Psi\rangle$ are given by

$$\log\langle\Psi|\Psi\rangle = \sum_{n=1}^{\infty} \log(1+s_n^2), \quad \langle N\rangle = 2\sum_{n=1}^{\infty} \frac{s_n^2}{1+s_n^2}.$$

$\frac{x}{1+x} \leq \log(1+x)$ implies $\langle N\rangle \leq 2\log\langle\Psi|\Psi\rangle$.

Since $g(x) = \frac{x}{1+x}(\log(1+x))^{-1}$ is decreasing for $x \geq 0$,

$$\langle N\rangle = 2\sum_{n=1}^{\infty} \frac{s_n^2}{1+s_n^2} \geq 2g(s_1^2)\sum_{n=1}^{\infty} \log(1+s_n^2) = 2g(s_1^2)\log\langle\Psi|\Psi\rangle.$$

Then we need to evaluate s_1 and $\langle\Psi|\Psi\rangle$.

Average number of particles

$|\Psi\rangle = \exp\left(i \int \frac{dz}{2\pi i} f_<(z) J_\geq(z)\right)|0\rangle$ and the BCH formula give

$$\langle\Psi|\Psi\rangle = \left|\frac{1 - e^{-2\varepsilon + i(x_2 - x_1)}}{1 - e^{-2\varepsilon}}\right|^{1/2} \quad \Rightarrow \quad \log\langle\Psi|\Psi\rangle \sim O\left(\log \frac{|x_1 - x_2|}{\varepsilon}\right).$$

For the largest singular value s_1,

$$s_1^2 \leq \sum_{n=1}^{\infty} s_n^2 = \operatorname{tr} D^\dagger D = \int_0^{2\pi} \frac{dx_1}{2\pi} \int_0^{2\pi} \frac{dx_2}{2\pi} \left|D_{<,<}(e^{ix_1}, e^{ix_2})\right|^2 \lesssim O\left(\left(\log \frac{1}{\varepsilon}\right)^2\right).$$

Because $g(x) = \frac{x}{1+x}(\log(1+x))^{-1}$, $g(s_1^2) \gtrsim O\left(\left(\log\log \frac{1}{\varepsilon}\right)^{-1}\right)$.

Average number of particles

We conclude that

$$O\left(\frac{\log \frac{1}{\varepsilon}}{\log \log \frac{1}{\varepsilon}}\right) \lesssim \langle N \rangle \lesssim O\left(\log \frac{|x_1 - x_2|}{\varepsilon}\right).$$

The lower bound diverges in the limit $\varepsilon \to 0$, so $\langle N \rangle$ also diverges.

We did a numerical calculation and confirmed that

$$\langle N \rangle \sim O\left(\log \frac{1}{\varepsilon}\right),$$

which suggests that $\langle N \rangle$ saturates the upper bound.

Summary of our results

- By unfolding, we can regard the ML boundary condition as the ML wall, which is a topological defect.
- The ML wall acts on the currents $J^i(x)$ by a matrix R^{ij} locally.
- We can compute the action of the ML wall on $\overline{\psi}^1(x_1)\psi^1(x_2)|0\rangle$ by writing it in terms of $J^i(x)$.
- The energy density and the charge density are still localized in the transformed state.
- The average number of particles $\langle N \rangle$ in the transformed state diverges in the limit $\varepsilon \to 0$ of perfectly localized wavepackets.

Future work: Application to the original 4d problem, lattice formulation.

Thank you for listening!

Backup slides

Monopole spherical harmonics

Let $\boldsymbol{A}$ be a vector potential for a monopole magnetic field $\boldsymbol{B} = \frac{g}{2}\frac{\boldsymbol{r}}{r^3}$. The Weyl operator for a left-handed Weyl fermion is

$$i\partial_t + \boldsymbol{\sigma} \cdot \boldsymbol{\pi} = i\partial_t + \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla} - iq\boldsymbol{A}),$$

which commutes with $\boldsymbol{J} = \boldsymbol{r} \times (-i\boldsymbol{\pi}) + \frac{1}{2}\boldsymbol{\sigma} - \frac{qg}{2}\hat{\boldsymbol{r}}$. The radial decomposition of the Weyl operator is

$$\boldsymbol{\sigma} \cdot \boldsymbol{\pi} = -i\sigma_r\left(\partial_r - \frac{1}{r}\boldsymbol{\sigma} \cdot (\boldsymbol{r} \times \boldsymbol{\pi})\right),$$

which acts on $\psi(t, r, \Omega) = \frac{1}{r}\chi(t, r, \Omega)$ as

$$\boldsymbol{\sigma} \cdot \boldsymbol{\pi}\left(\frac{1}{r}\chi\right) = \frac{1}{r}\left(-i\sigma_r\left(\partial_r - \frac{K}{r}\right)\chi\right), \quad K = \boldsymbol{\sigma} \cdot (\boldsymbol{r} \times \boldsymbol{\pi}) + 1.$$

Monopole spherical harmonics

Since J^2, J_z, K commute with each other, we can take eigenfunctions

$$J^2 \phi^{\lambda}_{j,\mu}(\Omega) = j(j+1)\phi^{\lambda}_{j,\mu}(\Omega),$$
$$J_z \phi^{\lambda}_{j,\mu}(\Omega) = \mu\phi^{\lambda}_{j,\mu}(\Omega),$$
$$K \phi^{\lambda}_{j,\mu}(\Omega) = \pm\nu\phi^{\lambda}_{j,\mu}(\Omega),$$

where $j = j_0, j_0+1, \ldots$, with $j_0 = \frac{|qg|-1}{2}$, $\mu = -j, -j+1, \ldots, j$ and $\nu = \sqrt{(j-j_0)(j+j_0+1)}$. Expand a fermion wavefunction as

$$\psi(t, r, \Omega) = \sum_{j,\mu} \frac{1}{r}\Big(f_{j,\mu}(t,r)\phi^{+}_{j,\mu}(\Omega) + g_{j,\mu}(t,r)\phi^{-}_{j,\mu}\Big),$$

and we set $\sigma_r \phi^{\pm}_{j,\mu} = \phi^{\mp}_{j,\mu}$

Monopole spherical harmonics

The action of the Weyl operator is then

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})\psi = \sum_{j,\mu} \frac{1}{r} \left[-i \left(\partial_r + \frac{\nu}{r} \right) g_{j,\mu} \phi^{+}_{j,\mu} - i \left(\partial_r - \frac{\nu}{r} \right) f_{j,\mu} \phi^{-}_{j,\mu} \right],$$

and the Weyl equation is reduced to the radial equation

$$i\partial_t f_{j,\mu} = i \left(\partial_r + \frac{\nu}{r} \right) g_{j,\mu}, \quad i\partial_t g_{j,\mu} = i \left(\partial_r - \frac{\nu}{r} \right) f_{j,\mu}.$$

For $j = j_0$, we can take $\phi^0_{j_0,\mu}$ to satisfy $\sigma_r \phi^0_{j_0,\mu} = \mathrm{sgn}(qg)\phi^0_{j_0,\mu}$. The Weyl equation becomes $i\partial_t f_{j,\mu} = i\,\mathrm{sgn}(qg)\partial_r f_{j,\mu}$. Thus if $qg > 0$, $f_{j,\mu}$ depends only on $t + r$, which is the free incoming 2d Weyl fermion.